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UNIFORM L^2-STABILITY FOR THE BOLTZMANN EQUATION

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ABSTRACT. We discuss a recent progress on the uniform L^2-stability for the Boltzmann equation in a close-to-Maxwellian regime.

1. INTRODUCTION

The purpose of this article is to present a recent formulation [6] on the uniform L^2-stability for the Boltzmann equation near a global Maxwellian. Consider the Boltzmann equation describing the phase space evolution of a distribution function $F = F(x, \xi, t)$ of moderately dilute gas particles with the physical position $x \in \Omega$ and the velocity $\xi \in \mathbb{R}^3$ at time $t \in \mathbb{R}_+$:

$$\begin{aligned} &\partial_t F + \xi \cdot \nabla_x F = Q(F, F), \quad x \in \Omega,\ \xi \in \mathbb{R}^3,\ t > 0, \\ &F(x, \xi, 0) = F^{in}(x, \xi), \end{aligned} \tag{1.1}$$

where $Q(F, F)$ is a quadratic collision operator whose explicit form will defined below.

Let (ξ', ξ'_*) be the post-collisional velocities defined in terms of pre-collisional velocities (ξ, ξ_*) and $\omega \in \mathbb{S}^2_+$:

$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega]\omega \quad \text{and} \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega]\omega. \tag{1.2}$$

In this case, the collision operator is given by the following form:

$$Q(F, F)(\xi) \equiv \frac{1}{\kappa} \iint_{\mathbb{R}^3 \times \mathbb{S}^2_+} q(\xi - \xi_*, \omega)(F'F'_* - FF_*) d\omega d\xi_*. \tag{1.3}$$

Here κ is the Knudsen number which is the ratio between the mean free path and the characteristic length of the flow, $\mathbb{S}^2_+ \equiv \{\omega \in \mathbb{S}^2 : (\xi - \xi_*) \cdot \omega > 0\}$, and we used standard abbreviated notations:

$$F' \equiv F(x, \xi', t), \quad F'_* \equiv F(x, \xi'_*, t), \quad F \equiv F(x, \xi, t) \quad \text{and} \quad F_* \equiv F(x, \xi_*, t).$$

We assume that the collision kernel $q(\cdot, \cdot)$ satisfies the inverse power law and the angular cut-off assumption:

$$q(\xi - \xi_*, \omega) = |\xi - \xi_*|^\gamma b_\gamma(\theta), \quad -\frac{3}{2} < \gamma \leq 1 \quad \text{and} \quad \frac{b_\gamma(\theta)}{\cos\theta} \leq b_* < \infty,$$

where θ is the angle between $\xi - \xi_*$ and ω:

$$\theta \equiv \cos^{-1}\Big(\frac{(\xi - \xi_*) \cdot \omega}{|\xi - \xi_*|}\Big).$$

The spatial domain Ω is assumed to be either whole space $\mathbb{R}^3$ or a torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{L}^3$ ($\mathbb{L}$: any 3-dimensional lattice in $\mathbb{R}^3$) to focus on the initial value problem. Throughout the paper, we shall restrict ourselves to the Boltzmann equation in a maxwellian regime, and denote by C the generic constant independent of time t.

In a global maxwellian regime, there are many literatures available for the existence theory of solutions and convergence toward a global maxwellian (see [2, 3] for a detailed survey). We next briefly review only the global existence theory of solutions to (1.1). In [10], Ukai first established the global existence of mild solutions to the Boltzmann equation for hard potential and hard sphere models combining a spectral analysis and a bootstrapping argument. Later Caflisch[1] and Ukai-Asano [11] further extended Ukai's seminal work to the moderately soft potentials $\gamma \in (-1,0]$ on a periodic domain and whole space respectively. For the general case of $\gamma \in (-3,0]$, the global existence of classical solutions was finally settled by Guo [5] employing an energy method. A global existence theory in an energy space $H^s_x(L^2_\xi)$ $(s \geq 8)$ became available only in recent years due to Liu-Yang-Yu [8] and Guo [4]. In particular, Liu, Yang and Yu in [7] introduced a macro-microscopic decomposition of the solution so that the Boltzmann equation can be rewritten as a new fluid type system and an equation for a non-fluid component. Hence the existence theory for (1.1) in a global Maxwellian regime is now in a good shape for small perturbations.

The rest of this paper is organized as follows. In Section 2, we review the basic properties of the linearized collision operator and micro-macro decomposition of a solution and the Boltzmann equation, and key trilinear estimates for the stability analysis. In Section 3, we discuss a priori uniform L^2-stability estimates [6] for the Boltzmann equation with moderately soft potentials $-\frac{3}{2} < \gamma \leq 1$.

Notations: Throughout the paper, we use various local and global norms on Ω, $\mathbb{R}^3_\xi$ and $\Omega \times \mathbb{R}^3_\xi$. Let $h = g(x,t,\xi)$ be a measurable function on $\Omega \times \mathbb{R}_t \times \mathbb{R}^3_\xi$. Below, $p, q \in [1,\infty]$:

$$\|h(x,t)\|_{L^q_\xi} \equiv \begin{cases} \left(\int_{\mathbb{R}^3} |f(x,\xi,t)|^q d\xi\right)^{\frac{1}{q}}, & 1 \leq q < \infty, \\ \mathrm{esssup}_{\xi \in \mathbb{R}^3} |f(x,\xi,t)|, & q = \infty, \end{cases}$$

$$\|h(t)\|_{L^p_x(L^q_\xi)} \equiv \begin{cases} \left(\int_{\mathbb{R}^3} \|h(x,t)\|^p_{L^q_\xi} dx\right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \mathrm{esssup}_{x \in \mathbb{R}^3} \|h(x,t)\|_{L^q_\xi}, & p = \infty, \end{cases} \qquad \|h(t)\|_{L^p} \equiv \|h(t)\|_{L^p_x(L^p_\xi)}.$$

2. Preliminaries

In this section, we review the basic properties of collision operators around a global Maxwellian, and micro-macro decomposition introduced in [7, 8]. Consider the Boltzmann equation

$$\begin{aligned} \partial_t F + \xi \cdot \nabla_x F &= Q(F,F), \qquad x \in \Omega,\ \xi \in \mathbb{R}^3,\ t \in \mathbb{R}_+, \\ F(0,x,\xi) &= F_0(x,\xi). \end{aligned}$$

We now introduce a symmetric bilinear operator $Q[F,G]$ associated with $Q(F,F)$:

$$Q[F,G](\xi) \equiv \frac{1}{2\kappa} \iint_{\mathbb{R}^3 \times \mathbb{S}^2_+} q(\xi - \xi_*, \omega)\Big(F' G'_* + F'_* G' - F G_* - F_* G\Big) d\omega d\xi_*.$$

Then it is easy to see that

$$Q[F,F] \equiv Q(F,F).$$

2.1. **The Boltzmann equation near M.** In this part, we study the linearization of the Boltzmann equation around a global Maxwellian. We first introduce the perturbation f as

$$F = M + M^{\frac{1}{2}} f, \qquad M \equiv \frac{1}{\sqrt{(2\pi)^3}} e^{-\frac{|\xi|^2}{2}}. \tag{2.1}$$

Then the perturbation f satisfies the linearized Boltzmann equation:

$$\partial_t f + \xi \cdot \nabla_x f = L(f) + \Gamma(f, f), \tag{2.2}$$

where $L(\cdot)$ and $\Gamma(\cdot,\cdot)$ are linear and nonlinear collision operators

$$L(f) \equiv 2M^{-\frac{1}{2}} Q[M, M^{\frac{1}{2}} f] \qquad \text{and} \qquad \Gamma(f, f) \equiv M^{-\frac{1}{2}} Q[M^{\frac{1}{2}} f, M^{\frac{1}{2}} f].$$

We formally define a quadratic form $\Gamma[\cdot,\cdot]$ associated with $\Gamma(\cdot,\cdot)$:

$$\Gamma[g, h] \equiv M^{-\frac{1}{2}} Q[M^{\frac{1}{2}} g, M^{\frac{1}{2}} h].$$

Proposition 2.1. [2] *For the Boltzmann equation* (2.2), *there exist positive constants $\nu_1 = \nu_1(\gamma), \nu_2 = \nu_2(\gamma), \sigma, k_1, k_2, k_3, k_4$ such that*

(1) *L has the decomposition*

$$L = -\nu(\xi) I + K,$$

where Id is an identity operator and $\nu(\xi)$ is a collision frequency satisfying

$$\nu_1 \langle \xi \rangle^\gamma \le \nu(\xi) \le \nu_2 \langle \xi \rangle^\gamma, \qquad \langle \xi \rangle = 1 + |\xi|, \quad \xi \in \mathbb{R}^3,$$

and K is a compact operator.

(2) *L is a non-positive self-adjoint operator on L^2_ξ with the estimate*

$$\langle Lh, h \rangle \le -\sigma \langle \nu^{\frac{1}{2}} \mathbf{P}_1 h, \mathbf{P}_1 h \rangle.$$

where $\langle \cdot,\cdot \rangle$ is a usual L^2-inner product.

2.2. **Micro-macro decomposition.** In this part, we briefly present the micro-macro decomposition which enable us to see the multi-scale nature of the Boltzmann equation. This beautiful idea of decompose the solution and the Boltzmann equation to see its corresponding fluid part and non-fluid part directly at a time was introduced by Liu and Yu in [7] to the study of the positivity of Boltzmann shock. This micro-macro decomposition will play a key role in our L^2-stability analysis for hard potential case in Section 3.2.

The linear collision operator L defines an unbounded symmetric operator on L^2_ξ:

$$L^2_\xi \equiv (L^2_\xi(\mathbb{R}^3), \langle \cdot,\cdot \rangle) \ \text{ and } \ \langle f, g \rangle \equiv \int_{\mathbb{R}^3} f(\xi) g(\xi) d\xi \qquad \text{for } f, g \in L^2_\xi.$$

The null space $\mathcal{N}$ of L is a five-dimensional vector space spanned by an orthonormal basis $\{\chi_i\}_{i=0}^{i=4}$:

$$\mathcal{N} \equiv \mathrm{span}\{\chi_0, \chi_1, \chi_2, \chi_3, \chi_4\},$$

$$\chi_0 = M^{\frac{1}{2}}, \quad \chi_i = \xi_i M^{\frac{1}{2}}, \quad \chi_4 = \frac{1}{\sqrt{6}} (|\xi|^2 - 3) M^{\frac{1}{2}}, \quad \langle \chi_i, \chi_j \rangle = \delta_{ij}, \qquad i = 1, 2, 3.$$

We decompose Hilbert space L^2_ξ as a direct sum of $\mathcal{N}$ and its orthogonal component $\mathcal{N}^\perp$, and we denote by $\mathbf{P}_0$ the projection on this null space and $\mathbf{P}_1$ the complementary projection:

$$\begin{cases} f = \mathbf{P}_0 f + \mathbf{P}_1 f = f_0 + f_1, \\ f_0 = \mathbf{P}_0 f \equiv \rho(x,t)\chi_0 + \sum_{i=1}^3 m_i(x,t)\chi_i + e(x,t)\chi_4, \\ \rho(x,t) = \langle f, \chi_0\rangle, \ m_i(x,t) = \langle f, \chi_i\rangle \ (i = 1,2,3), \ e(x,t) = \langle f, \chi_4\rangle, \\ f_1 = \mathbf{P}_1 f = f - f_0, \end{cases}$$

We next present trilinear estimates for nonlinear term $\Gamma[f+g, f-g](f-g)$. The property of $\Gamma[f+g, f-g] \in \mathcal{N}^\perp$ and Cauchy-Schawarz yield the following estimates.

Lemma 2.1. [6] *Let $-\frac{3}{2} \le \gamma \le 1$, and f, g be measurable functions in $\mathbb{R}^3_x \times \mathbb{R}^3_\xi$ satisfying*

$$||\nu^{\frac12}(f+g)||_{L^\infty_x(L^2_\xi)} < \infty, \quad ||f-g||_{L^2} + ||\nu^{\frac12}\mathbf{P}_1(f-g)||_{L^2} < \infty.$$

Then there exists a positive constant C independent of t such that

(i) $-\dfrac{3}{2} \le \gamma \le 0$;

$$\Big|\int_{\mathbb{R}^3} \langle \Gamma[f+g, f-g], f-g\rangle(x)dx\Big|$$
$$\le C\Big(||\nu^{\frac12} f(t)||^2_{L^\infty_x(L^2_\xi)} + ||\nu^{\frac12} g(t)||^2_{L^\infty_x(L^2_\xi)}\Big)||f(t)-g(t)||^2_{L^2} + \frac{\sigma}{2}||\nu^{\frac12}\mathbf{P}_1(f(t)-g(t))||^2_{L^2}.$$

(ii) $0 < \gamma \le 1$;

$$\Big|\int_{\mathbb{R}^3} \langle \Gamma[f+g, f-g], f-g\rangle(x)dx\Big|$$
$$\le C\Big(||\nu^{\frac12} f(t)||^2_{L^\infty_x(L^2_\xi)} + ||\nu^{\frac12} g(t)||^2_{L^\infty_x(L^2_\xi)}\Big)||f(t)-g(t)||^2_{L^2}$$
$$+\Big[C(||f(t)||_{L^\infty_x(L^2_\xi)} + ||g(t)||_{L^\infty_x(L^2_\xi)}) + \frac{\sigma}{2}\Big]||\nu^{\frac12}\mathbf{P}_1(f(t)-g(t))||^2_{L^2}.$$

3. A PRIORI UNIFORM L^2-STABILITY

In this section, we briefly present a priori uniform L^2-stability estimates. For details, we refer to [6]. Let f and g be two classical solutions to the Boltzmann equation (2.2) and $f, g \in L^\infty(\mathbb{R}_+; L^2_{x,\xi} \cap L^\infty_x(L^2_\xi))$. Then f and g satisfy

$$\partial_t f + \xi\cdot\nabla_x f = L(f) + \Gamma(f,f), \tag{3.1}$$

$$\partial_t g + \xi\cdot\nabla_x g = L(g) + \Gamma(g,g). \tag{3.2}$$

We subtract (3.2) from (3.1) and multiply $(f-g)$ to both sides to find

$$\partial_t|f-g|^2 + \xi\cdot\nabla_x|f-g|^2 = L(f-g)(f-g) + \Gamma[f+g, f-g](f-g). \tag{3.3}$$

We now integrate (3.3) with respect to (x,ξ) using the boundary condition and Proposition 2.1 to see

$$\begin{aligned} \frac{d}{dt}||f(t)-g(t)||^2_{L^2} &= \int_\Omega \langle L(f-g), f-g\rangle dx + \int_\Omega \langle \Gamma[f+g, f-g], f-g\rangle dx \\ &\le -\sigma||\nu^{\frac12}\mathbf{P}_1(f(t)-g(t))||_{L^2} + \Big|\int_\Omega \langle \Gamma[f+g, f-g], f-g\rangle dx\Big|. \end{aligned} \tag{3.4}$$

We set the uniform L^2-stability criterion as follows.

$$\int_0^\infty \Big(||\nu^{\frac12} f(s)||^2_{L^\infty_x(L^2_\xi)} + ||\nu^{\frac12} g(t)||^2_{L^\infty_x(L^2_\xi)}\Big)dt < \infty. \tag{3.5}$$

3.1. **Soft potential and Maxwellian molecule:** $-\frac{3}{2} < \gamma \leq 0$. Suppose two smooth perturbations f and g satisfy the stability condition (3.5). In (3.4), we use Lemma 2.1 to derive a Gronwall type inequality:

$$\begin{aligned}\frac{d}{dt}\|f(t)-g(t)\|_{L^2}^2 \leq & -\frac{\sigma}{2}\|\nu^{\frac{1}{2}}\mathbf{P}_1(f(t)-g(t))\|_{L^2}^2 \\ & + C\Big(\|\nu^{\frac{1}{2}}f(t)\|^2_{L^\infty_x(L^2_\xi)} + \|\nu^{\frac{1}{2}}g(t)\|^2_{L^\infty_x(L^2_\xi)}\Big)\|f(t)-g(t)\|^2_{L^2}.\end{aligned}$$

Then Gronwall's lemma yields

$$\begin{aligned}&\|f(t)-g(t)\|^2_{L^2} + \frac{\sigma}{2}\int_0^t \|\nu^{\frac{1}{2}}\mathbf{P}_1(f(s)-g(s))\|^2_{L^2}ds \\ &\quad\leq \exp\Big[C\int_0^t\Big(\|\nu^{\frac{1}{2}}f(s)\|^2_{L^\infty_x(L^2_\xi)} + \|\nu^{\frac{1}{2}}g(t)\|^2_{L^\infty_x(L^2_\xi)}\Big)dt\Big]\|f^{in}-g^{in}\|^2_{L^2} \\ &\quad\leq C\|f^{in}-g^{in}\|^2_{L^2}.\end{aligned}$$

This yields the uniform L^2-stability estimate.

Theorem 3.1. [6] *For $\gamma \in (-\frac{3}{2}, 0]$ and let F and G be two classical solutions to* (1.1) *in $L^\infty(\mathbb{R}^+; L^2(M^{-\frac{1}{2}}d\xi dx) \cap L^\infty_x(L^2(M^{-\frac{1}{2}}d\xi)))$ corresponding to initial data F^{in}, G^{in} respectively. Suppose the smooth perturbations f and g satisfy the condition* (3.5). *Then we have*

$$\sup_{0\leq t<\infty}\|F(t)-G(t)\|_{L^2(M^{-1/2}d\xi dx)} \leq C\|F^{in}-G^{in}\|_{L^2(M^{-1/2}d\xi dx)},$$

where C is a positive constant independent of t.

Remark 3.1. *As a direct application of the above theorem, the classical solutions in* [1, 5, 11] *are uniformly L^2-stable.*

3.2. **Hard potential and hard sphere model:** $0 < \gamma \leq 1$. Suppose two smooth perturbations f and g satisfy the stability condition (3.5) and the smallness condition:

$$\|f(t)\|_{L^\infty_x(L^2_\xi)} + \|g(t)\|_{L^\infty_x(L^2_\xi)} \ll \frac{\sigma}{4}. \tag{3.6}$$

In (3.4), we use Lemma 2.1 to get

$$\begin{aligned}\frac{d}{dt}\|f(t)-g(t)\|^2_{L^2} \leq\ & C\Big(\|\nu^{\frac{1}{2}}f(t)\|^2_{L^\infty_x(L^2_\xi)} + \|\nu^{\frac{1}{2}}g(t)\|^2_{L^\infty_x(L^2_\xi)}\Big)\|f(t)-g(t)\|^2_{L^2} \\ & + \Big[-\frac{\sigma}{2} + C\Big(\|f(t)\|^2_{L^\infty_x(L^2_\xi)} + \|g(t)\|^2_{L^\infty_x(L^2_\xi)}\Big)\Big]\|\nu^{\frac{1}{2}}\mathbf{P}_1(f(t)-g(t))\|^2_{L^2}.\end{aligned}\tag{3.7}$$

We use (3.7) to find

$$\begin{aligned}\frac{d}{dt}\|f(t)-g(t)\|^2_{L^2} \ \leq\ & C\Big(\|\nu^{\frac{1}{2}}f(t)\|^2_{L^\infty_x(L^2_\xi)} + \|\nu^{\frac{1}{2}}g(t)\|^2_{L^\infty_x(L^2_\xi)}\Big)\|f(t)-g(t)\|^2_{L^2} \\ & - \frac{\sigma}{4}\|\nu^{\frac{1}{2}}\mathbf{P}_1(f(t)-g(t))\|^2_{L^2}.\end{aligned}$$

Then Gronwall's lemma yield the following stability estimate.

Theorem 3.2. [6] *For $\gamma \in (0, 1]$ and let F and G be two small classical solutions to* (1.1) *in $L^\infty(\mathbb{R}^+; L^2(M^{-\frac{1}{2}}d\xi dx) \cap L^\infty_x(L^2(M^{-\frac{1}{2}}d\xi)))$ corresponding to small initial data F^{in}, G^{in}*

respectively. Suppose the smooth perturbations f and g satisfy (3.5) *and* (3.6). *Then we have*

$$\sup_{0\le t<\infty} \|F(t)-G(t)\|_{L^2(M^{-1/2}d\xi dx)} \le C\|F^{in}-G^{in}\|_{L^2(M^{-1/2}d\xi dx)},$$

where C is a positive constant independent of t.

Remark 3.2. *As a direct application of this theorem, the classical solutions in* [12] *are uniformly L^2-stable.*

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